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# ***On Orthogonal Substitutions that can be expressed as a Function of a Single Alternate (or Skew Symmetric) Linear Substitution.***

BY HENRY TABER.

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In Crelle's Journal, Vol. 32, Cayley gave an expression in terms of the minimum number of parameters for the general proper orthogonal substitution of  $N$  variables. Cayley's expression gives all proper orthogonal substitutions except those of which  $-1$  is a latent root (root of the characteristic equation). In a paper read August 25, 1893, at the Mathematical Congress in Chicago, it was shown by the author that the positive or negative square of Cayley's expression gives every real proper orthogonal substitution of any number of variables,\* and every imaginary proper orthogonal substitution of two, three, four, or six variables. Imaginary proper orthogonal substitutions of five, or of more than six variables, are not all given by the positive or negative square of Cayley's expression. Thus the proper orthogonal substitution of five variables, whose matrix is

$$\begin{pmatrix} 1, & 0, & 0, & 0, & 0, \\ 0, & -1 - \frac{1}{2}, & \frac{1}{2}\sqrt{-1}, & 0, & \sqrt{-1}, \\ 0, & \frac{1}{2}\sqrt{-1}, & -\frac{1}{2}, & 0, & 1, \\ 0, & 0, & 0, & -1, & 0, \\ 0, & -\sqrt{-1}, & -1, & 0, & -1, \end{pmatrix}$$

cannot be expressed in this way.

I find that the positive or negative square of Cayley's expression gives all orthogonal substitutions or matrices that can be expressed as a rational polynomial in powers of a single alternate or skew symmetric matrix.† In the paper

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\* Cf. Proc. Am. Acad. Arts and Scs., Vol. XXVI, p. 212.

† This includes not only all orthogonal substitutions that are rational integral functions of a single alternate or skew symmetric matrix, but also all orthogonal matrices which are rational functions of a single skew symmetric matrix (as those which are given by Cayley's expression), and all orthogonal matrices which can be expressed as a convergent power series in a single skew symmetric matrix. For the two latter classes of functions can be reduced to rational polynomials in powers of the skew symmetric matrix by the identical equation to that matrix.

above referred to, it was shown that every orthogonal matrix is given by the positive or negative square of Cayley's expression except those of which both  $\pm 1$  are latent roots.\* For such orthogonal matrices the theorem is therefore to be proved. The theorem may be proved as follows:

1. Let  $\phi$  be any orthogonal matrix whose distinct latent roots are  $g_0 = 1$ ,  $g_\epsilon = -1$ , and  $g_r, g_r^{-1}$ , for  $r = 1, 2, \dots, \mu$ . Corresponding, severally and respectively, to the distinct latent roots of  $\phi$  are certain rational integral functions of  $\phi$ , which I term the principal auxiliary functions of  $\phi$ . The principal auxiliary functions of  $\phi$  corresponding respectively to the distinct latent roots

$$+1, -1, g_r, g_r^{-1}, \quad (r = 1, 2, \dots, \mu),$$

will be denoted by

$$\Phi_0, \Phi_\epsilon, \Phi_r, \Phi_r', \quad (r = 1, 2, \dots, \mu).$$

These functions are formed as follows. If the identical equation to  $\phi$  is

$$(\phi - 1)^{m_0} (\phi + 1)^{m_\epsilon} (\phi - g_1)^{m_1} (\phi - g_1^{-1})^{m_1} \dots = 0,$$

and if

$$\begin{aligned} T_0(g_s) &= \frac{[(\phi - 1)^{m_0} - (g_s - 1)^{m_0}]^{m_\epsilon} [(\phi - 1)^{m_0} - (g_s^{-1} - 1)^{m_0}]^{m_s}}{[(g_s - 1)(g_s^{-1} - 1)]^{m_0 m_s}}, \\ T_\epsilon(g_s) &= \frac{[(\phi + 1)^{m_\epsilon} - (g_s + 1)^{m_\epsilon}]^{m_s} [(\phi + 1)^{m_\epsilon} - (g_s^{-1} + 1)^{m_\epsilon}]^{m_s}}{[(g_s + 1)(g_s^{-1} + 1)]^{m_\epsilon m_s}}, \\ T_r(g_s) &= \frac{[(\phi - g_r)^{m_r} - (g_s - g_r)^{m_r}]^{m_s} [(\phi - g_r)^{m_r} - (g_s^{-1} - g_r)^{m_r}]^{m_s}}{[(g_s - g_r)(g_s^{-1} - g_r)]^{m_r m_s}}, \end{aligned}$$

then

$$\Phi_0 = (-1)^{m_\epsilon(m_0+1)} \frac{[(\phi - 1)^{m_0} - (-1 - 1)^{m_0}]^{m_\epsilon}}{2^{m_0 m_\epsilon}} \Gamma_0(g_1) \Gamma_0(g_2) \dots \Gamma_0(g_\mu),$$

$$\Phi_\epsilon = (-1)^{m_0} \frac{[(\phi + 1)^{m_\epsilon} - (1 + 1)^{m_\epsilon}]^{m_0}}{2^{m_0 m_\epsilon}} \Gamma_\epsilon(g_1) \Gamma_\epsilon(g_2) \dots \Gamma_\epsilon(g_\mu),$$

and for  $r = 1, 2, \dots, \mu$ ,

$$\begin{aligned} \Phi_r &= (-1)^{m_0 + m_\epsilon(m_r+1)} \frac{[(\phi - g_r)^{m_r} - (1 - g_r)^{m_r}]^{m_0}}{(1 - g_r)^{m_r m_0}} \frac{[(\phi - g_r)^{m_r} - (-1 - g_r)^{m_r}]^{m_\epsilon}}{(1 + g_r)^{m_r m_\epsilon}} \\ &\quad \times \Gamma_r(g_1) \Gamma_r(g_2) \dots \Gamma_r(g_{r-1}) \Gamma_r(g_{r+1}) \dots \Gamma_r(g_\mu). \end{aligned}$$

The function  $\Phi_r'$  is obtained from  $\Phi_r$  by substituting throughout in the expression for the latter  $g_r^{-1}$  for  $g_r$ .

The binary products of different functions all vanish; moreover,

$$\Phi_0^2 = \Phi_0, \quad \Phi_\epsilon^2 = \Phi_\epsilon,$$

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\* The latent roots of a matrix are the roots of its characteristic equation. The term "latent root" is Sylvester's.

and for  $r = 1, 2, \dots, \mu$ ,

$$\Phi_r^2 = \Phi_r, \quad \Phi_r'^2 = \Phi_r'.$$

We also have

$$1 = \Phi_0 + \Phi_e + \Phi_1 + \Phi_1' + \Phi_2 + \Phi_2' + \dots$$

and

$$\begin{aligned} (\phi - 1)^{m_0} \Phi_0 &= 0, \\ (\phi + 1)^{m_e} \Phi_e &= 0, \\ (\phi - g_r)^{m_r} \Phi_r &= 0, \\ (\phi - g_r^{-1})^{m_r} \Phi_r' &= 0, \\ r &= 1, 2, \dots, \mu. \end{aligned}$$

The matrices  $\Phi_0$  and  $\Phi_e$  are symmetric, and  $\Phi_r'$  is the transverse (or conjugate) of  $\Phi_r$ .

2. Let  $\theta$  be an alternate or skew symmetric matrix; and let the distinct latent roots of  $\theta$  other than zero be  $h_r, h_{\nu+r} = -h_r$ , for  $r = 1, 2, \dots, \nu$ . The principal auxiliary function of  $\theta$  corresponding to  $h_r$  will be denoted by  $\Theta_r$ , and that corresponding to  $h_{\nu+r}$  by  $\Theta_{\nu+r}$ .

We have for  $r, s = 1, 2, \dots, \nu$ ,

$$\Theta_r \Theta_{\nu+s} = 0,$$

and if  $r \neq s$ ,

$$\Theta_r \Theta_s = 0.$$

Moreover,

$$\Theta_r^2 = \Theta_r, \quad \Theta_{\nu+r}^2 = \Theta_{\nu+r}, \quad (r = 1, 2, \dots, \nu).$$

The transverse of  $\Theta_r$  is  $\Theta_{\nu+r}$ .

If zero is a latent root of  $\theta$ , the principal auxiliary function of  $\theta$  corresponding to zero will be denoted by  $\Theta_0$ , and we have

$$\Theta_0 \Theta_r = 0, \quad \Theta_0 \Theta_{\nu+r} = 0, \quad (r = 1, 2, \dots, \nu)$$

and

$$\Theta_0^2 = \Theta_0.$$

The matrix  $\Theta_0$  is symmetric.

If zero is not a latent root of  $\theta$ , and if the identical equation to  $\theta$  is

$$\begin{aligned} &(\theta^2 - h_1^2)^{n_1} (\theta^2 - h_2^2)^{n_2} \dots (\theta^2 - h_\nu^2)^{n_\nu} \\ &= (\theta - h_1)^{n_1} (\theta - h_{\nu+1})^{n_1} (\theta - h_2)^{n_2} (\theta - h_{\nu+2})^{n_2} \dots (\theta - h_\nu)^{n_\nu} (\theta - h_{2\nu})^{n_\nu} = 0, \end{aligned}$$

then, for  $r = 1, 2, \dots, \nu$ ,

$$\Theta_r = \frac{[(\theta - h_r)^{n_r} - (-h_r - h_r)^{n_r}]}{(2h_r)^{n_r n_r}} G_r(h_1) G_r(h_2) \dots G_r(h_{r-1}) G_r(h_{r+1}) \dots G_r(h_\nu),$$

in which

$$G_r(h_s) = \frac{[(\theta - h_r)^{n_r} - (h_s - h_r)^{n_r}]^{n_s} [(\theta - h_r)^{n_r} - (-h_s - h_r)^{n_r}]^{n_s}}{(h_r^2 - h_s^2)^{n_r n_s}}.$$

On the other hand, if zero is a latent root of  $\theta$ , and if the identical equation to  $\theta$  is

$$\theta^{n_0} (\theta^2 - h_1^2)^{n_1} (\theta^2 - h_2^2)^{n_2} \dots (\theta^2 - h_\nu^2)^{n_\nu} = 0,$$

then

$$\Theta_r = (-1)^{n_0(n_r+1)} \frac{[(\theta - h_r)^{n_r} - (-h_r)^{n_r}]^{n_0}}{h_r^{n_r n_0}} \frac{[(\theta - h_r)^{n_r} - (-h_r - h_r)^{n_r}]^{n_r}}{(2h_r)^{n_r n_r}} \\ \times G_r(h_1) G_r(h_2) \dots G_r(h_{r-1}) G_r(h_{r+1}) \dots G_r(h_\nu).$$

And if

$$\mathbf{G}(h_s) = (-1)^{n_0 n_s} \frac{[\theta^{n_0} - h_s^{n_0}]^{n_s} [\theta^{n_0} - (-h_s)^{n_0}]^{n_s}}{h_s^{2n_0 n_s}},$$

then

$$\Theta_0 = \mathbf{G}(h_1) \mathbf{G}(h_2) \dots \mathbf{G}(h_\nu).$$

The function  $\Theta_{\nu+r}$  is obtained from  $\Theta_r$  by substituting throughout in the latter  $h_{\nu+r} = -h_r$  for  $h_r$ .

3. Let  $\phi$  be any function of  $\theta$  which, by means of the identical equation to  $\theta$ , can be expressed as a finite polynomial in integer powers of  $\theta$ ; thus, let

$$\phi = f(\theta) = A\theta^n + B\theta^{n-1} + \dots + K\theta + L.$$

Then

1). If  $g$  is a latent root of  $\phi$ , there is one or more latent roots of  $\theta$  satisfying the equation

$$f(x) - g = 0.$$

If  $h$  is a latent root of  $\theta$  and satisfies this equation, it may be said to belong to  $g$ .

2). If  $g$  and  $h$  are latent roots of  $\phi$  and  $\theta$  respectively, and if

$$f(h) - g = 0,$$

then

$$f(-h) - g^{-1} = 0.$$

That is, if  $h$  is a latent root of  $\theta$  belonging to the latent root  $g$  of  $\phi$ , then the latent root  $-h$  of  $\theta$  belongs to the latent root  $g^{-1}$  of  $\phi$ .

3). If  $g$  is a latent root of  $\phi$ , the multiplicity of  $g$  is equal to the sum of the multiplicities of the latent roots of  $\theta$  which satisfy the equation

$$f(x) - g = 0,$$

i. e. which belong to  $g$ .

4). If  $g$  is a latent root of  $\phi$ , and if  $h_a, h_\beta$ , etc., comprise all the latent roots of  $\theta$  which satisfy the equation

$$f(x) - g = 0,$$

then the principal auxiliary function of  $\phi$  corresponding to  $g$  is equal to the sum of the principal auxiliary functions of  $\theta$  corresponding to  $h_a, h_\beta$ , etc., namely, to

$$\Theta_a + \Theta_\beta + \dots$$

That is, if  $g$  is a latent root of  $\phi$ , the auxiliary function of  $\phi$  corresponding to  $g$  is equal to the sum of the principal auxiliary functions of  $\theta$  corresponding to the latent roots of  $\theta$  belonging to  $g$ .

4. From 2) it follows that if zero is a latent root of  $\theta$ , it belongs either to  $\pm 1$ , i. e. it is either a root of the equation

$$f(x) - 1 = 0,$$

or of

$$f(x) + 1 = 0.$$

Assume either, that zero is not a latent root of  $\theta$ , or if zero is a latent root of  $\theta$ , that  $f(0) = 1$  (i. e. that zero belongs to  $+1$ ), and let  $h_a, h_\beta$ , etc., be roots of the equation  $f(x) + 1 = 0$ ; then by 2)  $-h_a, -h_\beta$ , etc., are also roots of this equation. If, therefore,  $\alpha', \beta'$ , etc., denote, respectively, the smallest positive residue (modulus  $\nu$ ) of  $\alpha, \beta$ , etc., by 4),

$$\Phi_\epsilon = \Theta_{\alpha'} + \Theta_{\nu+\alpha'} + \Theta_{\beta'} + \Theta_{\nu+\beta'} + \dots$$

5. In the paper by the author above referred to it was shown that if

$$\begin{aligned} \psi_1 = & [1 + c_1(\phi - 1) + c_2(\phi - 1)^2 + \dots + c_{m_0-1}(\phi - 1)^{m_0-1}] \Phi_0 \\ & + \sqrt{-1} [1 + c_1\left(\frac{\phi+1}{-1}\right) + c_2\left(\frac{\phi+1}{-1}\right)^2 + \dots + c_{m_\epsilon-1}\left(\frac{\phi+1}{-1}\right)^{m_\epsilon-1}] \Phi_\epsilon \\ & + \sum_1^\mu \sqrt{g_r} [1 + c_1\left(\frac{\phi-g_r}{g_r}\right) + c_2\left(\frac{\phi-g_r}{g_r}\right)^2 + \dots + c_{m_r-1}\left(\frac{\phi-g_r}{g_r}\right)^{m_r-1}] \Phi_r \\ & + \sum_1^\mu \frac{1}{\sqrt{g_r}} [1 + c_1\left(\frac{\phi-g_r^{-1}}{g_r^{-1}}\right) + c_2\left(\frac{\phi-g_r^{-1}}{g_r^{-1}}\right)^2 + \dots + c_{m_r-1}\left(\frac{\phi-g_r^{-1}}{g_r^{-1}}\right)^{m_r-1}] \Phi'_r, \end{aligned}$$

in which 1,  $c_1, c_2$ , etc., denote the coefficients of unity, the first, second, and successive powers of  $x$  in the development by the binomial theorem of  $(1+x)^\dagger$ , then

$$\phi = \psi_1^2;$$

and denoting the transverse of  $\psi_1$  by  $\overline{\psi}_1$ ,

$$\overline{\psi}_1 \psi_1 = 1 - 2\Phi_\epsilon.$$

Let

$$\psi = \psi_1 (1 - 2\Theta_{\nu+\alpha'} - 2\Theta_{\nu+\beta'} - \dots),$$

then

$$\begin{aligned} \psi^2 &= \psi_1^2 (1 - 2\Theta_{\nu+\alpha'} - 2\Theta_{\nu+\beta'} - \dots)^2, \\ &= \psi_1^2 \\ &= \phi; \end{aligned}$$

and

$$\begin{aligned} \overline{\psi} \psi &= \overline{\psi}_1 \psi_1 (1 - 2\Theta_{\alpha'} - 2\Theta_{\beta'} - \dots)(1 - 2\Theta_{\nu+\alpha'} - 2\Theta_{\nu+\beta'} - \dots) \\ &= (1 - 2\Phi_\epsilon)^2 \\ &= 1. \end{aligned}$$

Therefore  $\psi$  is orthogonal.

6. We have

$$\begin{aligned} (\psi - 1)^{m_0} \Phi_0 &= (\psi_1 - 1)^{m_0} \Phi_0 = 0, \\ (\psi^2 + 1)^{m_\epsilon} \Phi_\epsilon &= (\phi + 1)^{m_\epsilon} \Phi_\epsilon = 0, \end{aligned}$$

and for  $r = 1, 2, \dots, \mu$ ,

$$\begin{aligned} (\psi - \sqrt{g_r})^{m_r} \Phi_r &= (\psi_1 - \sqrt{g_r})^{m_r} \Phi_r = 0, \\ \left(\psi - \frac{1}{\sqrt{g_r}}\right)^{m_r} \Phi'_r &= \left(\psi_1 - \frac{1}{\sqrt{g_r}}\right)^{m_r} \Phi'_r = 0. \end{aligned}$$

Consequently

$$\begin{aligned} (\psi - 1)^{m_0} &= (\psi - 1)^{m_0} (1 - \Phi_0), \\ (\psi^2 + 1)^{m_\epsilon} &= (\psi^2 + 1)^{m_\epsilon} (1 - \Phi_\epsilon), \end{aligned}$$

and for  $r = 1, 2, \dots, \mu$ ,

$$\begin{aligned} (\psi - \sqrt{g_r})^{m_r} &= (\psi - \sqrt{g_r})^{m_r} (1 - \Phi_r), \\ \left(\psi - \frac{1}{\sqrt{g_r}}\right)^{m_r} &= (\psi - \sqrt{g_r})^{m_r} (1 - \Phi'_r); \end{aligned}$$

and therefore

$$\begin{aligned} &(\psi - 1)^{m_0} (\psi^2 + 1)^{m_\epsilon} (\psi - \sqrt{g_1})^{m_1} \left(\psi - \frac{1}{\sqrt{g_1}}\right)^{m_1} \dots \\ &= \left[ (\psi - 1)^{m_0} (\psi^2 + 1)^{m_\epsilon} (\psi - \sqrt{g_1})^{m_1} \left(\psi - \frac{1}{\sqrt{g_1}}\right)^{m_1} \dots \right] [1 - \Phi_0 - \Phi_\epsilon - \Phi_1 - \Phi'_1 - \dots] = 0. \end{aligned}$$

Whence it follows that  $-1$  is not a latent root of  $\psi$ . But then since  $|\psi + 1| \neq 0$ , we may put

$$\Upsilon = \frac{1 - \psi}{1 + \psi};$$

whence we obtain

$$\Upsilon = \frac{1 - \Psi}{1 + \Psi} = \frac{1 - \Psi^{-1}}{1 + \Psi^{-1}} = \frac{\Psi - 1}{\Psi + 1} = -\Upsilon$$

and

$$\Psi = \frac{1 - \Upsilon^*}{1 + \Upsilon}.$$

Therefore

$$\Phi = \Psi^2 = \left( \frac{1 - \Upsilon}{1 + \Upsilon} \right)^2.$$

7. On the other hand, if zero is a latent root of  $\Phi$  and belongs to  $-1$ , i. e. if

$$f(0) + 1 = 0,$$

let  $\Phi = -\phi$ , and  $F(x) = -f(x)$ . Then

$$\Phi = F(\theta)$$

and

$$F(0) - 1 = 0,$$

i. e. zero belongs to the latent root  $+1$  of  $\Phi$ . Consequently, for a proper choice of  $\Upsilon$  we may put

$$\Phi = \left( \frac{1 - \Upsilon}{1 + \Upsilon} \right)^2,$$

i. e.

$$\phi = -\left( \frac{1 - \Upsilon}{1 + \Upsilon} \right)^2.$$

8. Let  $\phi = f(\theta)$  be an orthogonal substitution of  $N$  variables. If  $N$  is odd, zero is a latent root of  $\theta$  of odd multiplicity; therefore, by 3), if  $|\phi| = +1$ , zero belongs to the latent root  $+1$  of  $\phi$ , and consequently

$$\phi = \left( \frac{1 - \Upsilon}{1 + \Upsilon} \right)^2.$$

If  $N$  is odd and  $|\phi| = -1$ , then  $-\phi$  is a proper orthogonal matrix, and we have

$$\phi = -\left( \frac{1 - \Upsilon}{1 + \Upsilon} \right)^2.$$

If  $N$  is even, since the determinant of both the positive and negative square of Cayley's expression is then equal to  $+1$ ,  $\phi = f(\theta)$  is a proper orthogonal substitution. Therefore, no improper orthogonal substitution in an even number of variables can be expressed as a function of a single skew symmetric matrix.

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\* This is Cayley's expression for an orthogonal substitution in terms of a skew symmetric or alternate linear substitution.

9. Let  $\phi$  be an orthogonal substitution given by the positive square of Cayley's expression;  $\phi$  is then a proper orthogonal substitution. If  $-1$  is a latent root of  $\phi$ , then

$$\left| \frac{\Upsilon^2 + 1}{(1 + \Upsilon)^2} \right| = |\phi + 1| = 0,$$

therefore,  $|\Upsilon \pm \sqrt{-1}| = 0$ . But if  $\sqrt{-1}$  is a latent root of  $\Upsilon$ ,  $-\sqrt{-1}$  is also a latent root, since  $\Upsilon$  is skew symmetric. If the nullity\* of  $\Upsilon + \sqrt{-1}$  is  $m$  (i. e. if the minors determinants of  $\phi$  of order  $N - m + 1$  all vanish, but not all the minors of order  $N - m$ ), the nullity of  $\Upsilon - \sqrt{-1}$  is also  $m$ . Therefore by the corollary of the law of nullity,† the nullity of  $\Upsilon^2 + 1$  and consequently of  $\phi + 1$  is  $2m$ . Whence it follows that if  $\phi$  is any proper orthogonal substitution of  $N$  variables given by the positive square of Cayley's expression, and if the  $2m^{\text{th}}$  minors of  $\phi + 1$  (the minors of order  $N - 2m$ ) all vanish, then the  $(2m + 1)^{\text{th}}$  minors all vanish also. In other words, if  $\phi$  is a proper orthogonal substitution in  $N$  variables given by the square of Cayley's expression, the minor of  $\phi + 1$  of highest order that does not vanish is of even or odd order according as  $N$  is even or odd. Further, the nullity of  $(\phi + 1)^2$  is even, i. e. the non-vanishing minors of  $(\phi + 1)^2$  of highest order are of  $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$  order if  $N$  is  $\begin{Bmatrix} \text{even} \\ \text{odd} \end{Bmatrix}$ .

Similarly with respect to  $(\phi + 1)^3$  and the successive powers of  $\phi + 1$ .

Conversely, I find that all proper orthogonal substitutions that satisfy these conditions, and of which the multiplicity of the latent root  $-1$  is 0, 2, 4 or 6, are given by the positive square of Cayley's expression,‡ and probably this is true for any proper orthogonal substitution satisfying the above conditions.

I also find that the positive square of Cayley's expression gives all proper orthogonal substitutions  $\phi$  whose fundamental syzygy§ contains  $\phi + 1$  to the first power only, and all which satisfy the above conditions and whose fundamental syzygy contains  $\phi + 1$  to the second power only.

CLARK UNIVERSITY, WORCESTER, MASS., December 8, 189 .

\* This term is Sylvester's. Nullity of order  $m$  is equivalent to *rank (Rang)*  $N - m$ .

† See Johns Hopkins Univ. Circs., Vol. III, p. 10.

‡ Evidently all proper orthogonal substitutions of which  $-1$  is not a latent root satisfy these conditions; and from Stieltje's theorem it follows that all those of which  $-1$  is a latent root of multiplicity two also satisfy the conditions. But all proper orthogonal substitutions of which  $-1$  is a latent root of multiplicity greater than two do not satisfy the conditions, and are not therefore all given by the square of Cayley's expression, e. g. the orthogonal substitution of  $p$ .

§ The rational integral function of  $\phi$  of lowest order that vanishes I term the fundamental syzygy in  $\phi$ .